

# Explicit evaluation of quadratic Euler sums

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**Abstract** In this paper, we work out some explicit formulae for double nonlinear Euler sums involving harmonic numbers and alternating harmonic numbers. As applications of these formulae, we give new closed form representations of several quadratic Euler sums through Riemann zeta function and linear sums. The given representations are new.

**Keywords** Polylogarithm function; Euler sum; Riemann zeta function.

**AMS Subject Classifications (2010):** 11M06; 11M32; 33B15

## 1 Introduction

In this paper, harmonic numbers, alternating harmonic numbers and their generalizations define as

$$H_n = \sum_{j=1}^n \frac{1}{j}, \quad \zeta_n(k) = \sum_{j=1}^n \frac{1}{j^k}, \quad L_n(k) = \sum_{j=1}^n \frac{(-1)^{j-1}}{j^k}, \quad 1 \leq k \in \mathbb{Z}. \quad (1.1)$$

The classical linear Euler sum is defined by

$$S_{p,q} = \sum_{n=1}^{\infty} \frac{1}{n^q} \sum_{k=1}^n \frac{1}{k^p}, \quad (1.2)$$

where  $p, q$  are positive integers,  $q \geq 2$ . In 1742, Goldbach proposed to Euler the problem of expressing the  $S_{p,q}$  in terms of values at positive integers of the Riemann zeta function  $\zeta(s)$ . The Riemann zeta function and alternating Riemann zeta function are defined respectively by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1; \quad \bar{\zeta}(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \Re(s) \geq 1.$$

Euler showed this problem in the case  $p = 1$  and gave a general formula for odd weight  $p + q$  in 1775. He conjectured that the double linear sums would be reducible to zeta values when  $p + q$  is odd, and even gave what he hoped to obtain the general formula. In [3], D. Borwein, J.M. Borwein and R. Girgensohn proved conjecture and formula, and in [2], D.H. Bailey, J.M. Borwein and R. Girgensohn conjectured that the double linear sums when  $p + q > 7$ ,  $p + q$  is even, are not reducible.

Let  $\pi = (\pi_1, \dots, \pi_k)$  be a partition of integer  $p$  and  $p = \pi_1 + \dots + \pi_k$  with  $\pi_1 \leq \pi_2 \leq \dots \leq \pi_k$ .

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The classical double nonlinear Euler sum of index  $\pi, q$  is defined as follows (see [12])

$$S_{\pi, q} = \sum_{n=1}^{\infty} \frac{\zeta_n(\pi_1) \zeta_n(\pi_2) \cdots \zeta_n(\pi_k)}{n^q}, \quad (1.3)$$

where the quantity  $\pi_1 + \cdots + \pi_k + q$  and  $k$  called the weight and the degree.

In general, we can also define the double Euler sums by the series

$$\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{m_1} \zeta_n^{q_i}(k_i) \prod_{j=1}^{m_2} L_n^{l_j}(h_j)}{n^p}, \quad \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{m_1} \zeta_n^{q_i}(k_i) \prod_{j=1}^{m_2} L_n^{l_j}(h_j) (-1)^{n-1}}{n^p}, \quad (1.4)$$

where  $p(p > 1), m_1, m_2, q_i, k_i, h_j, l_j$  are positive integers. If  $\sum_{i=1}^{m_1} (k_i q_i) + \sum_{j=1}^{m_2} (h_j l_j) + p = C$  ( $C$  is a positive integer), then we call the identity  $C$ th-order Euler sums.

In [12], Philippe Flajolet and Bruno Salvy gave explicit reductions to zeta values for all linear sums

$$\sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^q}, \quad \sum_{n=1}^{\infty} \frac{L_n(p)}{n^q}, \quad \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^q} (-1)^{n-1}, \quad \sum_{n=1}^{\infty} \frac{L_n(p)}{n^q} (-1)^{n-1}$$

when  $p + q$  is an odd weight. The evaluation of linear sums in terms of values of Riemann zeta function and polylogarithm function at positive integers is known when  $(p, q) = (1, 3), (2, 2)$ , or  $p + q$  is odd [6, 12, 18]. For instance, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n}{n^3} (-1)^{n-1} &= -2\text{Li}_4\left(\frac{1}{2}\right) + \frac{11}{4}\zeta(4) + \frac{1}{2}\zeta(2)\ln^2 2 - \frac{1}{12}\ln^4 2 - \frac{7}{4}\zeta(3)\ln 2, \\ \sum_{n=1}^{\infty} \frac{L_n(1)}{n^3} (-1)^{n-1} &= \frac{3}{2}\zeta(4) + \frac{1}{2}\zeta(2)\ln^2 2 - \frac{1}{12}\ln^4 2 - 2\text{Li}_4\left(\frac{1}{2}\right). \end{aligned}$$

The polylogarithm function defined for  $|x| \leq 1$  by

$$\text{Li}_p(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^p}, \quad \Re(p) > 1. \quad (1.5)$$

The relationship between the values of the Riemann zeta function and Euler sums has been studied by many authors, for example see [2-4, 6-17, 19-20].

From [18], for a multi-index  $S = (s_1, s_2, \dots, s_k)$  ( $s_i \in \mathbb{N}$ ,  $s_1 \geq 2$ ), the multiple zeta value (MZV for short) and the multiple zeta-star value (MZSV for short) are defined, respectively, by convergent series

$$\begin{aligned} \zeta(S) &= \zeta(s_1, s_2, \dots, s_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}, \\ \zeta^*(S) &= \zeta(s_1, s_2, \dots, s_k) = \sum_{n_1 \geq n_2 \geq \dots \geq n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}. \end{aligned}$$

We put a bar on top of  $s_j$  ( $j = 1, \dots, k$ ) if there is a sign  $(-1)^{n_j}$  appearing in the denominator on the right. For example,

$$\zeta^*(\bar{2}, 3) = \sum_{m \geq n \geq 1} \frac{(-1)^m}{m^2 n^3} = \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} \sum_{n=1}^m \frac{1}{n^3}.$$

By using the above notations, we have

$$\zeta^*(q, p) = S_{p,q} = \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^q}, \quad \zeta^*(\bar{q}, p) = - \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^q} (-1)^{n-1},$$

$$\zeta^*(q, \bar{p}) = - \sum_{n=1}^{\infty} \frac{L_n(p)}{n^q}, \quad \zeta^*(\bar{q}, \bar{p}) = \sum_{n=1}^{\infty} \frac{L_n(p)}{n^q} (-1)^{n-1}.$$

The main purpose of this paper is to evaluate some quadratic Euler sums which involving harmonic numbers and alternating harmonic numbers, either linearly or nonlinearly. In this paper, we will prove that all quadratic double sums

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2m}} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{2m}} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{2m}} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{2m}}, \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{2m}}$$

are reducible to polynomials in zeta values and to linear sums,  $m$  is a positive integer. Similarly, we have the following relations

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2m}} (-1)^{n-1} = -2\zeta^*(\overline{2m}, 1, 1) + \zeta^*(\overline{2m}, 2), \quad \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{2m}} (-1)^{n-1} = -2\zeta^*(\overline{2m}, \bar{1}, \bar{1}) + \zeta^*(\overline{2m}, 2),$$

$$\sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{2m}} = 2\zeta^*(2m, \bar{1}, \bar{1}) - \zeta^*(2m, 2), \quad \sum_{n=1}^{\infty} \frac{\zeta_n^2(p)}{n^{2m}} = 2\zeta^*(2m, p, p) - \zeta^*(2m, 2p),$$

$$\sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{2m}} = -\zeta^*(2m, \bar{1}, 1) - \zeta^*(2m, 1, \bar{1}) + \zeta^*(2m, \bar{2}),$$

$$\sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{2m}} (-1)^{n-1} = \zeta^*(\overline{2m}, \bar{1}, 1) + \zeta^*(\overline{2m}, 1, \bar{1}) - \zeta^*(\overline{2m}, \bar{2}).$$

The following lemma will be useful in the development of the main theorems.

**Lemma 1.1** *Let  $n$  be a positive integers. We have the following relations*

$$\sum_{k=1}^{n-1} \frac{L_k(1)}{n-k} (-1)^{k-1} = 2(-1)^{n-1} \sum_{k=1}^n \frac{H_k}{k} (-1)^{k-1} - 2(-1)^{n-1} L_n(2), \quad (1.6)$$

$$\sum_{m=n+1}^{\infty} \frac{L_m(1)}{m-n} (-1)^{m-n} = \sum_{k=1}^n \frac{H_k}{k} (-1)^{k-1} - \sum_{j=1}^{\infty} \frac{L_j(1)}{j} (-1)^{j-1}, \quad (1.7)$$

$$(-1)^{n-1} \sum_{k=1}^{n-1} \frac{L_k(1)}{n-k} = \sum_{k=1}^{n-1} \frac{H_k}{n-k} (-1)^k, \quad (1.8)$$

$$\sum_{m=n+1}^{\infty} \frac{H_m}{m-n} (-1)^{m-n} = \frac{L_n^2(1) + \zeta_n(2)}{2} - \ln 2 (H_n + L_n(1)) - \sum_{n=1}^{\infty} \frac{H_j}{j} (-1)^{j-1}. \quad (1.9)$$

where  $\sum_{n=1}^{\infty} \frac{H_n(-1)^{n-1}}{n} = \frac{\zeta(2) - \ln^2 2}{2}$ ,  $\sum_{n=1}^{\infty} \frac{L_n(1)(-1)^{n-1}}{n} = \frac{\zeta(2) + \ln^2 2}{2}$ .

*Proof.* To prove (1.6), using Cauchy product formula, we have

$$\frac{\ln(1+x)}{1-x} = \left( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \right) \left( \sum_{n=1}^{\infty} x^{n-1} \right) = \sum_{n=1}^{\infty} L_n(1) x^n. \quad (1.10)$$

Furthermore, by (1.10) and Cauchy product formula, we can obtain

$$\frac{\ln^2(1+x)}{1-x} = \sum_{n=1}^{\infty} (-1)^{n+1} \left( \sum_{k=1}^n \frac{L_k(1)}{n-k+1} (-1)^{k-1} \right) x^{n+1}. \quad (1.11)$$

On the other hand, from [14], we get

$$\ln^2(1+x) = 2 \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} x^n - \sum_{n=1}^{\infty} \frac{H_n}{n} (-1)^{n-1} x^n \right\}. \quad (1.12)$$

Similarly, applying (1.12), then have

$$\frac{\ln^2(1+x)}{1-x} = 2 \sum_{n=1}^{\infty} \sum_{k=1}^n \left\{ L_n(2) - \frac{H_k}{k} (-1)^{k-1} \right\} x^n. \quad (1.13)$$

Comparing (1.11) and (1.13), we can obtain (1.6). Now, we prove (1.7). We see that we may rewrite the left side of (1.7) as

$$\begin{aligned} \sum_{m=n+1}^{\infty} \frac{L_m(1)}{m-n} (-1)^{m-n} &= \sum_{j=1}^{\infty} \frac{L_{n+j}(1)}{j} (-1)^j \\ &= \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \left\{ \sum_{k=1}^j \frac{(-1)^{k-1}}{k} + \sum_{k=j+1}^{n+j} \frac{(-1)^{k-1}}{k} \right\} \\ &= \sum_{j=1}^{\infty} \frac{L_j(1)}{j} (-1)^j + \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \sum_{j=1}^{\infty} \left( \frac{1}{j} - \frac{1}{j+k} \right) \\ &= \sum_{k=1}^n \frac{H_k}{k} (-1)^{k-1} - \sum_{j=1}^{\infty} \frac{L_j(1)}{j} (-1)^{j-1}. \end{aligned} \quad (1.14)$$

Similarly to the proof of (1.6) and (1.7), we can prove (1.8) and (1.9).

## 2 Main Theorems and Proof

In this section, we will establish some explicit relationships which involve alternating quadratic Euler sums and linear sums.

**Theorem 2.1** *For integer  $m \geq 1$ , we have*

$$\sum_{j=1}^{m-1} (-1)^{j-1} \bar{\zeta}(m-j+1) \zeta^*(j+1, 1) + (-1)^{m-1} \ln 2 \zeta^*(m+1, 1) - \zeta^*(\overline{m+2}, 1)$$

$$\begin{aligned}
& -(-1)^{m-1} \ln 2 \zeta^* (\overline{m+1}, 1) - (-1)^{m-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} H_n L_n(1)}{n^{m+1}} \\
& = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} H_n^2}{n^{m+1}} - \frac{1}{2} \zeta^* (\overline{m+1}, 2) + \zeta(2) \bar{\zeta}(m+1).
\end{aligned} \tag{2.1}$$

*Proof.* Let

$$I_m = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^m (n+k)}.$$

Using the definition of  $I_m$ , we can easily verify that

$$I_m = \sum_{j=1}^{m-1} \frac{(-1)^{j-1}}{k^j} \bar{\zeta}(m-j+1) + \frac{(-1)^{m-1}}{k^m} \ln 2 + \frac{(-1)^{m+k}}{k^m} \ln 2 - \frac{(-1)^{m+k}}{k^m} L_k(1). \tag{2.2}$$

On the other hand, we have

$$\sum_{k=1}^{\infty} \frac{H_k}{k(n+k)} = \frac{1}{n} \left( \frac{1}{2} H_n^2 + \frac{1}{2} \zeta_n(2) + \zeta(2) - \frac{H_n}{n} \right). \tag{2.3}$$

Multiplying (2.3) by  $\frac{(-1)^{n-1}}{n^m}$  and summing with respect to  $n$ . The result is

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{n-1} H_k}{k n^m (n+k)} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} H_n^2}{n^{m+1}} - \frac{1}{2} \zeta^* (\overline{m+1}, 2) + \zeta(2) \bar{\zeta}(m+1) + \zeta^* (\overline{m+2}, 1). \tag{2.4}$$

Substituting (2.2) into (2.4) yields desired result.

**Theorem 2.2** *Let  $s \geq 1$  be integer. Then the following identity holds*

$$\begin{aligned}
\frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^2}{n^s} (-1)^{n-1} - \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^s} (-1)^{n-1} &= -\frac{3}{2} \zeta^* (\bar{s}, 2) - s \zeta^* (\overline{s+1}, 1) - \zeta^* (s+1, 1) \\
&\quad - \sum_{j=2}^s \zeta^* (j, 1) \bar{\zeta}(s+1-j) + \ln 2 \zeta^* (\bar{s}, 1),
\end{aligned} \tag{2.5}$$

$$\frac{3}{2} \sum_{n=1}^{\infty} \frac{H_n^2 - \zeta_n(2)}{n^s} = (s+1) \zeta^* (s+1, 1) - \sum_{j=2}^{s-1} \zeta^* (j, 1) \zeta(s+1-j), \quad 2 \leq s \in \mathbb{Z}. \tag{2.6}$$

*Proof.* For real  $-1 \leq x < 1$  and integers  $s$  ( $s \geq 2$ ) and  $t \geq 0$ , consider function

$$\begin{aligned}
T_{s,t}(x, y) &= \sum_{\substack{n,m=1 \\ n \neq m}}^{\infty} \frac{H_n x^n y^m}{n^s m^t (m-n)} = \sum_{\substack{n,m=1 \\ n \neq m}}^{\infty} \frac{H_n x^n y^m (m-n+n)}{n^s m^{t+1} (m-n)} \\
&= \sum_{\substack{n,m=1 \\ n \neq m}}^{\infty} \frac{H_n x^n y^m}{n^s m^{t+1}} + \sum_{\substack{n,m=1 \\ n \neq m}}^{\infty} \frac{H_n x^n y^m}{n^{s-1} m^{t+1} (m-n)}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{H_n x^n}{n^s} \left( \sum_{m=1}^{\infty} \frac{y^m}{m^{t+1}} - \frac{y^n}{n^{t+1}} \right) + T_{s-1,t+1}(x, y) \\
&= \left( \sum_{n=1}^{\infty} \frac{H_n x^n}{n^s} \right) Li_{t+1}(y) - \left( \sum_{n=1}^{\infty} \frac{H_n x^n y^n}{n^{s+t+1}} \right) + T_{s-1,t+1}(x, y). \tag{2.7}
\end{aligned}$$

Define

$$H_s^{(k)}(x) = \sum_{n=1}^{\infty} \frac{H_n^k x^n}{n^s}, \quad -1 \leq x < 1, \quad 0 \leq k \in \mathbb{Z}, \quad \Re(s) \geq 1.$$

Obviously,  $H_s^{(0)}(x) = Li_s(x)$ . Hence, we can rewrite (2.7) as

$$T_{s,t}(x, y) = H_s^{(1)}(x) H_{t+1}^{(0)}(y) - H_{s+t+1}^{(1)}(xy) + T_{s-1,t+1}(x, y). \tag{2.8}$$

Telescoping this gives

$$T_{s,t}(x, y) = \sum_{j=1}^s H_j^{(1)}(x) H_{s+t+1-j}^{(0)}(y) - s H_{s+t+1}^{(1)}(xy) + T_{0,s+t}(x, y). \tag{2.9}$$

Taking  $t = 0$  in (2.9) yield

$$T_{s,0}(x, y) = \sum_{j=1}^s H_j^{(1)}(x) H_{s+1-j}^{(0)}(y) - s H_{s+1}^{(1)}(xy) + T_{0,s}(x, y). \tag{2.10}$$

From the definition of  $T_{s,t}(x, y)$ , we conclude that

$$\begin{aligned}
T_{s,0}(x, y) &= \sum_{\substack{n,m=1 \\ n \neq m}}^{\infty} \frac{H_n x^n y^m}{n^s (m-n)} = \sum_{n=1}^{\infty} \frac{H_n x^n}{n^s} \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{y^m}{m-n} \\
&= \sum_{n=1}^{\infty} \frac{H_n x^n y^n}{n^s} \sum_{m=n+1}^{\infty} \frac{y^{m-n}}{m-n} - \sum_{n=1}^{\infty} \frac{H_n x^n}{n^s} \sum_{m=1}^{n-1} \frac{y^m}{n-m} \\
&= H_s^{(1)}(xy) H_1^{(0)}(y) - \sum_{n=1}^{\infty} \frac{H_n x^n}{n^s} \left( \sum_{j=1}^{n-1} \frac{y^{n-j}}{j} \right). \tag{2.11}
\end{aligned}$$

$$\begin{aligned}
T_{0,s}(x, y) &= \sum_{\substack{n,m=1 \\ n \neq m}}^{\infty} \frac{H_n x^n y^m}{m^s (m-n)} = \sum_{m=1}^{\infty} \frac{y^m}{m^s} \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{H_n x^n}{m-n} \\
&= - \sum_{m=1}^{\infty} \frac{x^m y^m}{m^s} \sum_{n=m+1}^{\infty} \frac{H_n x^{n-m}}{n-m} + \sum_{m=1}^{\infty} \frac{y^m}{m^s} \sum_{n=1}^{m-1} \frac{H_n x^n}{m-n}. \tag{2.12}
\end{aligned}$$

Noting that

$$\sum_{n=m+1}^{\infty} \frac{H_n x^{n-m}}{n-m} = \sum_{j=1}^{\infty} \frac{H_{n+j} x^j}{j} x^j = \sum_{j=1}^{\infty} \frac{x^j}{j} \left( \sum_{k=1}^j \frac{1}{k} + \sum_{k=j+1}^{n+j} \frac{1}{k} \right)$$

$$\begin{aligned}
&= \sum_{j=1}^{\infty} \frac{H_j}{j} x^j + \sum_{j=1}^{\infty} \frac{x^j}{j} \sum_{k=1}^n \frac{1}{k+j} \\
&= H_1^{(1)}(x) + \sum_{k=1}^n \frac{1}{k} \sum_{j=1}^{\infty} x^j \left( \frac{1}{j} - \frac{1}{k+j} \right),
\end{aligned} \tag{2.13}$$

$$\sum_{j=1}^{n-1} \frac{H_j}{n-j} = H_n^2 - \zeta_n(2), \quad \sum_{k=1}^n \frac{H_k}{k} = \frac{H_n^2 + \zeta_n(2)}{2}. \tag{2.14}$$

$$\lim_{x \rightarrow 1, y \rightarrow 1} \left\{ H_s^{(1)}(xy) H_1^{(0)}(y) - H_s^{(1)}(x) H_1^{(0)}(y) \right\} = 0, \tag{2.15}$$

$$\lim_{x \rightarrow 1, y \rightarrow 1} \left\{ H_s^{(0)}(xy) H_1^{(1)}(x) - H_s^{(0)}(y) H_1^{(1)}(x) \right\} = 0, \tag{2.16}$$

$$\lim_{x \rightarrow 1, y \rightarrow -1} \left\{ H_s^{(0)}(xy) H_1^{(1)}(x) - H_s^{(0)}(y) H_1^{(1)}(x) \right\} = 0. \tag{2.17}$$

Combining (2.10)-(2.17) and taking  $(x, y) \rightarrow (1, 1), (1, -1)$ , we can obtain (2.5) and (2.6). If taking  $(x, y) \rightarrow (1, -1)$  in (2.10)-(2.12) and combining (1.9), we can give the following Corollary.

**Corollary 2.3** *Let  $s \geq 2$  be integer, we have*

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n-1} \frac{H_k}{n-k} (-1)^k}{n^s} &= \sum_{n=1}^{\infty} \frac{H_n^2}{n^s} (-1)^{n-1} + \frac{1}{2} (\zeta(2) - \ln^2 2) \bar{\zeta}(s) + \ln 2 (\zeta^*(\bar{s}, \bar{1}) - \zeta^*(\bar{s}, 1)) \\
&\quad - \sum_{j=1}^{s-1} \zeta^*(\bar{j}, 1) \zeta(s+1-j) + (s+1) \zeta^*(\overline{s+1}, 1) \\
&\quad - \frac{1}{2} \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^s} (-1)^{n-1} + \frac{1}{2} \zeta^*(\bar{s}, 2).
\end{aligned} \tag{2.18}$$

In the same way, we can obtain the following Theorem.

**Theorem 2.4** *Let  $s \geq 1$  be integer, then have*

$$\begin{aligned}
\zeta^*(s, \bar{1}, 1) &= - \sum_{n=1}^{\infty} \frac{\sum_{k=1}^n \frac{H_k}{k} (-1)^{k-1}}{n^s} \\
&= s \zeta^*(s+1, \bar{1}) - \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^s} + 2 \zeta^*(s, \bar{2}) - \ln 2 \zeta^*(s, \bar{1}) \\
&\quad + \zeta^*(\overline{s+1}, \bar{1}) + \zeta(s) \zeta^*(\bar{1}, \bar{1}) + \sum_{j=1}^s \zeta^*(\bar{j}, \bar{1}) \bar{\zeta}(s+1-j), \quad (s \geq 2).
\end{aligned} \tag{2.19}$$

$$\zeta^*(\bar{s}, \bar{1}, 1) = \sum_{n=1}^{\infty} \frac{\sum_{k=1}^n \frac{H_k}{k} (-1)^{k-1}}{n^s} (-1)^{n-1}$$

$$\begin{aligned}
&= (s+1) \zeta^\star(\overline{s+1}, \overline{1}) + 2\zeta^\star(\overline{s}, \overline{2}) - \bar{\zeta}(s) \zeta^\star(\overline{1}, \overline{1}) - \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^s} (-1)^{n-1} \\
&\quad - \sum_{j=1}^{s-1} \zeta^\star(\overline{j}, \overline{1}) \bar{\zeta}(s+1-j). \tag{2.20}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sum_{k=1}^{n-1} \frac{L_k(1)}{n-k}}{n^s} &= s\zeta^\star(\overline{s+1}, \overline{1}) + \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^s} (-1)^{n-1} + \zeta^\star(s+1, \overline{1}) \\
&\quad + \sum_{j=2}^s \zeta^\star(j, \overline{1}) \bar{\zeta}(s+1-j) - \ln 2 \zeta^\star(\overline{s}, \overline{1}) \\
&\quad - \ln 2 (\zeta^\star(\overline{s}, \overline{1}) - \zeta^\star(\overline{s}, 1)) + \zeta^\star(\overline{s+1}, 1, \overline{1}). \tag{2.21}
\end{aligned}$$

*Proof.* Similarly to the proof of Theorem 2.2, considering function

$$T_{s,t}(x, y) = \sum_{\substack{n,m=1 \\ n \neq m}}^{\infty} \frac{L_n(1) x^n y^m}{n^s m^t (m-n)}. \tag{2.22}$$

We have the following identity

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{L_n(1)}{n^s} x^n y^n \sum_{n=1}^{\infty} \frac{y^n}{n} - \sum_{n=1}^{\infty} \frac{L_n(1)}{n^s} x^n \sum_{j=1}^{n-1} \frac{y^{n-j}}{j} \\
&\quad + \sum_{n=1}^{\infty} \frac{x^n y^n}{n^s} \sum_{m=n+1}^{\infty} \frac{L_m(1) x^{m-n}}{m-n} - \sum_{n=1}^{\infty} \frac{y^n}{n^s} \sum_{m=1}^{n-1} \frac{L_m(1) x^m}{n-m} \\
&= \sum_{j=1}^s \left( \sum_{n=1}^{\infty} \frac{L_n(1)}{n^j} x^n \right) \left( \sum_{n=1}^{\infty} \frac{y^n}{n^{s+1-j}} \right) - s \sum_{n=1}^{\infty} \frac{L_n(1)}{n^{s+1}} x^n y^n. \tag{2.23}
\end{aligned}$$

Noting that

$$\begin{aligned}
\sum_{m=n+1}^{\infty} \frac{L_m(1)}{m-n} x^{m-n} &= \sum_{j=1}^{\infty} \frac{L_{n+j}(1)}{j} x^j = \sum_{j=1}^{\infty} \frac{x^j}{j} \left\{ \sum_{k=1}^j \frac{(-1)^{k-1}}{k} + \sum_{k=j+1}^{n+j} \frac{(-1)^{k-1}}{k} \right\} \\
&= \sum_{j=1}^{\infty} \frac{L_j(1)}{j} x^j + \sum_{k=1}^n (-1)^{k-1} \sum_{j=1}^{\infty} \frac{(-x)^j}{j(k+j)} \\
&= \sum_{j=1}^{\infty} \frac{L_j(1)}{j} x^j + \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \sum_{j=1}^{\infty} (-x)^j \left( \frac{1}{j} - \frac{1}{k+j} \right) \tag{2.24}
\end{aligned}$$

and

$$\lim_{x \rightarrow 1} \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \sum_{j=1}^{\infty} (-x)^j \left( \frac{1}{j} - \frac{1}{k+j} \right) = \sum_{k=1}^n \frac{L_k(1)}{k} - \ln 2 (H_n + L_n(1)). \tag{2.25}$$

Taking  $(x, y) \rightarrow (-1, -1), (-1, 1), (1, -1)$  and combining (1.6)(1.7)(2.23)(2.24), we can obtain (2.19)-(2.21). Next, we give two explicit relationships which involve alternating double Euler sums and triple linear sums.



**Theorem 2.5** For  $p \geq 1$  and  $p \in \mathbb{Z}$ , we have

$$\begin{aligned}
\zeta^\star(\overline{p+1}, 1, \overline{1}) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{p+1}} \left( \sum_{k=1}^n \frac{L_k(1)}{k} \right) \\
&= \sum_{i=1}^{p-1} (-1)^{i-1} \zeta(p+1-i) \bar{\zeta}(i+2) - (-1)^{p-1} \zeta^\star(\overline{p+2}, 1) \\
&\quad + \ln 2 (\zeta^\star(\overline{p+1}, \overline{1}) - \zeta^\star(\overline{p+1}, 1)) - \frac{1}{2} \ln^2 2 \{ \zeta(p+1) + \bar{\zeta}(p+1) \} \\
&\quad + \sum_{i=1}^{p-1} (-1)^{i-1} \zeta(p+1-i) \zeta^\star(\overline{i+1}, 1) \\
&\quad - (-1)^{p-1} \sum_{n=1}^{\infty} \frac{H_n^2}{n^{p+1}} (-1)^{n-1}. \tag{2.26}
\end{aligned}$$

$$\begin{aligned}
-\zeta^\star(p+1, 1, \overline{1}) &= \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \left( \sum_{k=1}^n \frac{L_k(1)}{k} \right) \\
&= \ln 2 (\zeta^\star(p+1, 1) - \zeta^\star(p+1, \overline{1})) + \sum_{i=1}^{p-1} (-1)^{i-1} \bar{\zeta}(p+1-i) \zeta^\star(\overline{i+1}, 1) \\
&\quad + \ln 2 (-1)^{p-1} \{ \zeta(p+2) + \bar{\zeta}(p+2) \} - (-1)^p \zeta^\star(p+2, \overline{1}) \\
&\quad - \frac{1}{2} \ln^2 2 \{ \zeta(p+1) + \bar{\zeta}(p+1) \} - \ln 2 (-1)^{p-1} (\zeta^\star(p+1, 1) - \zeta^\star(\overline{p+1}, 1)) \\
&\quad + \sum_{i=1}^{p-1} (-1)^{i-1} \bar{\zeta}(p+1-i) \bar{\zeta}(i+2) - (-1)^p \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{p+1}}. \tag{2.27}
\end{aligned}$$

*Proof.* Using integration by parts, we can verify that

$$\int_0^x t^{n-1} \text{Li}_q(t) dt = \sum_{i=1}^{q-1} (-1)^{i-1} \frac{x^n}{n^i} \text{Li}_{q+1-i}(x) + \frac{(-1)^q}{n^q} \ln(1-x) (x^n - 1) - \frac{(-1)^q}{n^q} \left( \sum_{k=1}^n \frac{x^k}{k} \right), \tag{2.28}$$

$$\int_0^x t^{n-1} \ln^2(1-t) dt = \frac{1}{n} (x^n - 1) \ln^2(1-x) - \frac{2}{n} \sum_{k=1}^n \frac{1}{k} \left\{ x^k \ln(1-x) - \sum_{j=1}^k \frac{x^j}{j} - \ln(1-x) \right\}. \tag{2.29}$$

Now, we consider the following integral

$$\int_0^1 \frac{\text{Li}_p(x) \ln^2(1+x)}{x} dx, \quad 1 \leq p \in \mathbb{Z}.$$

Using (1.12) and (2.28), we can find that

$$\int_0^1 \frac{\text{Li}_p(x) \ln^2(1+x)}{x} dx = 2 \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n-1}}{n^2} - \frac{H_n}{n} (-1)^{n-1} \right\} \int_0^1 x^{n-1} \text{Li}_p(x) dx$$

$$\begin{aligned}
&= 2 \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n-1}}{n^2} - \frac{H_n}{n} (-1)^{n-1} \right\} \left\{ \sum_{i=1}^{p-1} \frac{(-1)^{i-1}}{n^i} \zeta(p+1-i) + \frac{(-1)^{p-1}}{n^p} H_n \right\} \\
&= 2 \sum_{i=1}^{p-1} (-1)^{i-1} \zeta(p+1-i) \bar{\zeta}(i+2) - 2(-1)^{p-1} \zeta^*(\overline{p+2}, 1) \\
&\quad + 2 \sum_{i=1}^{p-1} (-1)^{i-1} \zeta(p+1-i) \zeta^*(\overline{i+1}, 1) - 2(-1)^{p-1} \sum_{n=1}^{\infty} \frac{H_n^2}{n^{p+1}} (-1)^{n-1}. \tag{2.30}
\end{aligned}$$

On the other hand, by (2.29), we have

$$\begin{aligned}
&\int_0^1 \frac{\text{Li}_p(x) \ln^2(1+x)}{x} dx = \sum_{n=1}^{\infty} \frac{1}{n^p} \int_0^1 x^{n-1} \ln^2(1+x) dx \\
&= \ln^2 2 \{ \zeta(p+1) + \bar{\zeta}(p+1) \} - 2 \ln 2 \{ \zeta^*(\overline{p+1}, 1) - \zeta^*(\overline{p+1}, 1) \} \\
&\quad + 2 \zeta^*(\overline{p+1}, 1, \bar{1}). \tag{2.31}
\end{aligned}$$

Combining (2.30) and (2.31), we can obtain (2.26). Similarly to the proof of (2.26), considering integral

$$\int_0^{-1} \frac{\text{Li}_p(x) \ln^2(1-x)}{x} dx,$$

we deduce (2.27) holds.

### 3 Representation of Euler sums by zeta values and linear sums

In this section, we consider the analytic representations of quadratic Euler sums of the form

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2m}} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{2m}} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{2m}} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{2m}}, \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{2m}}$$

through zeta values and linear sums, and give explicit formulae for several 6th-order quadratic sums in terms of zeta values and linear sums.

**Theorem 3.1** *For  $m \geq 1$  and  $m \in \mathbb{Z}$ , we have*

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2}{n^{2m}} (-1)^{n-1} &= -\zeta^*(\overline{2m}, 2) - (2m+1) \zeta^*(\overline{2m+1}, 1) - \zeta^*(2m+1, 1) \\
&\quad - \zeta(2) \bar{\zeta}(2m) - 2 \sum_{l=2}^m \zeta^*(2l-1, 1) \bar{\zeta}(2m+2-2l), \tag{3.1}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{2m}} (-1)^{n-1} &= \sum_{l=1}^{m-1} \zeta^*(2l, 1) \bar{\zeta}(2m+1-2l) - \ln 2 \zeta^*(\overline{2m}, 1) \\
&\quad + \ln 2 \zeta^*(2m, 1) + \frac{1}{2} \zeta^*(2m+1, 1) + \zeta^*(\overline{2m}, 2) \\
&\quad + \frac{2m-1}{2} \zeta^*(\overline{2m+1}, 1) - \frac{1}{2} \zeta(2) \bar{\zeta}(2m). \tag{3.2}
\end{aligned}$$

*Proof.* Replacing  $m$  by  $2m - 1$  in (2.1) and taking  $s = 2m$  in (2.5), we can obtain (3.1) and (3.2).

**Theorem 3.2** For  $p \geq 1$  and  $p \in \mathbb{Z}$ , we have

$$\begin{aligned}
& \frac{3}{2} \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{p+1}} (-1)^{n-1} - \left(1 + (-1)^{p-1}\right) \sum_{n=1}^{\infty} \frac{H_n^2}{n^{p+1}} (-1)^{n-1} \\
&= \ln 2 \left( \zeta^* (\overline{p+1}, \overline{1}) - \zeta^* (\overline{p+1}, 1) \right) + \frac{1}{2} \zeta(2) \{ \zeta(p+1) + \bar{\zeta}(p+1) \} \\
&+ \frac{1}{2} \zeta^* (\overline{p+1}, 2) - \sum_{j=2}^p \left( 1 - (-1)^{j-1} \right) \zeta^* (\bar{j}, 1) \zeta(p+2-j) \\
&- \sum_{j=2}^{p+1} \zeta^* (j, \overline{1}) \bar{\zeta}(p+2-j) + \left( p+2 + (-1)^{p-1} \right) \zeta^* (\overline{p+2}, 1) - \zeta^* (p+2, \overline{1}) \\
&- \sum_{i=1}^{p-1} (-1)^{i-1} \zeta(p+1-i) \bar{\zeta}(i+2) - (p+1) \zeta^* (\overline{p+2}, \overline{1}) + \ln 2 \zeta^* (\overline{p+1}, \overline{1}). \tag{3.3}
\end{aligned}$$

*Proof.* From (1.8)(2.18) and (2.21), we obtain

$$\begin{aligned}
\zeta^* (\bar{s}, 1, \overline{1}) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \left( \sum_{k=1}^n \frac{L_k(1)}{k} \right) \\
&= \sum_{n=1}^{\infty} \frac{H_n^2}{n^s} (-1)^{n-1} + 2 \ln 2 \left( \zeta^* (\bar{s}, \overline{1}) - \zeta^* (\bar{s}, 1) \right) - \frac{3}{2} \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^s} (-1)^{n-1} \\
&- \sum_{j=1}^{s-1} \zeta^* (\bar{j}, 1) \zeta(s+1-j) - \sum_{j=2}^s \zeta^* (j, \overline{1}) \bar{\zeta}(s+1-j) \\
&+ (s+1) \zeta^* (\overline{s+1}, 1) - s \zeta^* (\overline{s+1}, \overline{1}) + \ln 2 \zeta^* (\bar{s}, \overline{1}) \\
&- \zeta^* (s+1, \overline{1}) + \frac{1}{2} \zeta^* (\bar{s}, 2) + \frac{1}{2} \{ \zeta(2) - \ln^2 2 \} \bar{\zeta}(s). \tag{3.4}
\end{aligned}$$

Taking  $s = p+1$  in (3.4) and combining (2.26), we can obtain (3.3).

**Theorem 3.3** For  $p \geq 1$  and  $p \in \mathbb{Z}$ , we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{p+1}} + \left( (-1)^{p-1} - 1 \right) \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{p+1}} \\
&= \zeta^* (p+1, \overline{2}) - \ln 2 \zeta^* (p+1, \overline{1}) + \zeta^* (\overline{p+2}, \overline{1}) + \frac{1}{2} \zeta(2) \zeta(p+1) \\
&+ \sum_{i=1}^{p+1} \zeta^* (\bar{i}, \overline{1}) \bar{\zeta}(p+2-i) + (p+1 + (-1)^p) \zeta^* (p+2, \overline{1}) \\
&+ \frac{1}{2} \ln^2 2 \{ 2 \zeta(p+1) + \bar{\zeta}(p+1) \} + \ln 2 (-1)^{p-1} \left( \zeta^* (p+1, 1) - \zeta^* (\overline{p+1}, 1) \right) \\
&- \sum_{i=1}^{p-1} (-1)^{i-1} \bar{\zeta}(p+1-i) \zeta^* (\overline{i+1}, 1) - \ln 2 \left( \zeta^* (p+1, 1) - \zeta^* (p+1, \overline{1}) \right)
\end{aligned}$$

$$-\sum_{i=1}^{p-1} (-1)^{i-1} \bar{\zeta}(p+1-i) \bar{\zeta}(i+2) - \ln 2 (-1)^{p-1} \{ \zeta(p+2) + \bar{\zeta}(p+2) \}. \quad (3.5)$$

*Proof.* The following identity is easily derived

$$\sum_{k=1}^n \frac{H_k}{k} (-1)^{k-1} + \sum_{k=1}^n \frac{L_k(1)}{k} = H_n L_n(1) + L_n(2). \quad (3.6)$$

Multiplying (3.6) by  $\frac{1}{n^{p+1}}$  and summing with respect to  $n$ , the result is

$$\begin{aligned} \zeta^*(p+1, \bar{1}, 1) + \zeta^*(p+1, 1, \bar{1}) &= - \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \left( \sum_{k=1}^n \frac{H_k}{k} (-1)^{k-1} \right) - \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \left( \sum_{k=1}^n \frac{L_k(1)}{k} \right) \\ &= - \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{p+1}} + \zeta^*(p+1, \bar{2}). \end{aligned} \quad (3.7)$$

Substituting (2.19)(2.27) into (3.7), we deduce Theorem 3.3 holds.

**Theorem 3.4** For  $l \geq 2, p \geq 1$  and  $l, p \in \mathbb{Z}$ , we have

$$\sum_{n=1}^{\infty} \frac{\zeta_n^2(p)}{n^l} = 2^{2p+l-3} \left\{ \sum_{n=1}^{\infty} \frac{(\zeta_n(p) - L_n(p))^2}{n^l} \left( 1 - (-1)^{n-1} \right) \right\}, \quad (3.8)$$

$$\sum_{n=1}^{\infty} \frac{\zeta_n^2(p)}{(2n+1)^l} = 2^{2p-3} \left\{ \sum_{n=1}^{\infty} \frac{(\zeta_n(p) - L_n(p))^2}{n^l} \left( 1 + (-1)^{n-1} \right) \right\}. \quad (3.9)$$

*Proof.* By the definitions of harmonic numbers and alternating harmonic numbers, we have the relations

$$\zeta_{2n}(p) - L_{2n}(p) = \frac{1}{2^{p-1}} \zeta_n(p), \quad \zeta_{2n+1}(p) - L_{2n+1}(p) = \frac{1}{2^{p-1}} \zeta_n(p), \quad (3.10)$$

$$\zeta_{2n}(p) + L_{2n}(p) = 2h_n(p), \quad \zeta_{2n+1}(p) + L_{2n+1}(p) = 2h_{n+1}(p). \quad (3.11)$$

Here  $h_n(p)$  is defined by

$$h_n(p) = \sum_{j=1}^n \frac{1}{(2j-1)^p}.$$

Hence, from (3.10) and (3.11), we get

$$\sum_{n=1}^{\infty} \frac{(\zeta_n(p) - L_n(p))^2}{n^l} = \frac{1}{2^{2p+l-2}} \sum_{n=1}^{\infty} \frac{\zeta_n^2(p)}{n^l} + \frac{1}{2^{2p-2}} \sum_{n=1}^{\infty} \frac{\zeta_n^2(p)}{(2n+1)^l}, \quad (3.12)$$

$$\sum_{n=1}^{\infty} \frac{(\zeta_n(p) - L_n(p))^2}{n^l} (-1)^{n-1} = \frac{1}{2^{2p-2}} \sum_{n=1}^{\infty} \frac{\zeta_n^2(p)}{(2n+1)^l} - \frac{1}{2^{2p+l-2}} \sum_{n=1}^{\infty} \frac{\zeta_n^2(p)}{n^l}. \quad (3.13)$$

Combining (3.12) and (3.13), we can obtain (3.8) and (3.9). Taking  $p = 1, l = 2m$  in (3.8) and (3.9), we have

$$\sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{2m}} = \left( \frac{1}{2} - \frac{1}{2^{2m}} \right) \sum_{n=1}^{\infty} \frac{H_n^2}{n^{2m}} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{2m}} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^2 - 2H_n L_n(1) + L_n^2(1)}{n^{2m}} (-1)^{n-1}, \quad (3.14)$$

$$\sum_{n=1}^{\infty} \frac{H_n^2}{(2n+1)^{2m}} = \frac{1}{2} \left\{ \sum_{n=1}^{\infty} \frac{H_n^2 - 2H_n L_n(1) + L_n^2(1)}{n^{2m}} \left(1 + (-1)^{n-1}\right) \right\}. \quad (3.15)$$

From (2.6)(3.1)-(3.3)(3.5)(3.14) and (3.15), we have the following Theorem.

**Theorem 3.5** *For  $m \geq 1$  and  $m \in \mathbb{Z}$ , the quadratic sums*

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{2m}} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{2m}}, \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^{2m}} (-1)^{n-1} \\ & \sum_{n=1}^{\infty} \frac{H_n^2}{n^{2m}}, \sum_{n=1}^{\infty} \frac{H_n^2}{n^{2m}} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^{2m}}, \sum_{n=1}^{\infty} \frac{H_n^2}{(2n+1)^{2m}} \end{aligned}$$

*are reducible to linear sums.*

Next, we give some explicit formulae for several 4th and 6th-order quadratic sums in terms of zeta values and linear sums.

**Example 1** *Some illustrative examples follow.*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^2} &= \frac{43}{16} \zeta(4) + \frac{3}{4} \zeta(2) \ln^2 2 - \frac{1}{8} \ln^4 2 - 3\text{Li}_4\left(\frac{1}{2}\right), \\ \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^2} &= -\frac{13}{8} \zeta(4) + \frac{5}{2} \zeta(2) \ln^2 2 + \frac{1}{12} \ln^4 2 + 2\text{Li}_4\left(\frac{1}{2}\right), \\ \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^2} (-1)^{n-1} &= \frac{29}{16} \zeta(4) + \frac{3}{4} \zeta(2) \ln^2 2 - 3\text{Li}_4\left(\frac{1}{2}\right) - \frac{1}{8} \ln^4 2, \\ \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} (-1)^{n-1} &= \frac{41}{16} \zeta(4) + \frac{1}{2} \zeta(2) \ln^2 2 - \frac{1}{12} \ln^4 2 - \frac{7}{4} \zeta(3) \ln 2 - 2\text{Li}_4\left(\frac{1}{2}\right), \\ \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^2} (-1)^{n-1} &= -\frac{41}{16} \zeta(4) + 2\zeta(2) \ln^2 2 + \frac{1}{6} \ln^4 2 + \frac{7}{4} \zeta(3) \ln 2 + 4\text{Li}_4\left(\frac{1}{2}\right), \\ \sum_{n=1}^{\infty} \frac{H_n^2}{(2n+1)^2} &= -\frac{61}{16} \zeta(4) + \frac{7}{2} \zeta(2) \ln^2 2 + \frac{1}{3} \ln^4 2 + 8\text{Li}_4\left(\frac{1}{2}\right). \end{aligned}$$

Using mathematica, the numerical values of 6th-order quadratic sums and 6th-order linear sums, to 30 decimal digits, are:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n}{n^5} (-1)^{n-1} &= -\zeta^*(\bar{5}, 1) = 0.959151942504318157165421137321 \dots \\ \sum_{n=1}^{\infty} \frac{L_n(1)}{n^5} &= -\zeta^*(5, \bar{1}) = 1.02005194570145237930331996837 \dots \\ \sum_{n=1}^{\infty} \frac{L_n(1)}{n^5} (-1)^{n-1} &= \zeta^*(\bar{5}, \bar{1}) = 0.987441426403299713771650007985 \dots \\ \sum_{n=1}^{\infty} \frac{\zeta_n(2)}{n^4} (-1)^{n-1} &= -\zeta^*(\bar{4}, 2) = 0.934707899349253255197542851216 \dots \end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{L_n(2)}{n^4} &= -\zeta^*(4, \bar{2}) = 1.06358224101814909880154833539 \dots \\
\sum_{n=1}^{\infty} \frac{H_n^2}{n^4} (-1)^{n-1} &= 0.889343140860204925167721149031 \dots \\
\sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^4} (-1)^{n-1} &= 0.992525582596587194230080495889 \dots \\
\sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^4} &= 1.02740836872025737362228676589 \dots \\
\sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^4} &= 1.07692404770840527955071456146 \dots \\
\sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^4} (-1)^{n-1} &= 0.969581748999225091836494168654 \dots \\
\sum_{n=1}^{\infty} \frac{L_n(1) \zeta_n(2)}{n^3} &= 1.15935334356951415975457027807 \dots
\end{aligned}$$

From Theorem 3.1-3.4, we find that the 6th-order quadratic sums satisfies a relation involving homogeneous combinations of  $\zeta(2), \zeta(3), \zeta(4), \zeta(5), \zeta(6), \ln 2$  and 6th-order linear sums.

### Example 2

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2}{n^4} (-1)^{n-1} &= -\frac{175}{32} \zeta(6) + \frac{1}{2} \zeta^2(3) - \zeta^*(\bar{4}, 2) - 5\zeta^*(\bar{5}, 1), \\
\sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^4} &= \frac{15}{4} \ln^2 2 \zeta(4) + \frac{9}{4} \zeta(2) \zeta(3) \ln 2 - \frac{31}{4} \zeta(5) \ln 2 + \frac{231}{64} \zeta(6) - \frac{45}{32} \zeta^2(3) \\
&\quad + \zeta^*(\bar{5}, \bar{1}) + \zeta^*(4, \bar{2}), \\
\sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^4} (-1)^{n-1} &= \frac{15}{4} \ln^2 2 \zeta(4) + \frac{9}{4} \zeta(2) \zeta(3) \ln 2 - \frac{93}{16} \zeta(5) \ln 2 + \frac{35}{64} \zeta(6) - \frac{15}{16} \zeta^2(3) \\
&\quad - \zeta^*(\bar{4}, 2), \\
\sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^4} &= -\frac{3}{2} \zeta(2) \zeta(3) \ln 2 + \frac{31}{8} \zeta(5) \ln 2 + \frac{2359}{384} \zeta(6) - \frac{7}{64} \zeta^2(3) + \frac{1}{2} \zeta^*(\bar{5}, \bar{1}) \\
&\quad + \frac{1}{2} \zeta^*(4, \bar{2}) + 2\zeta^*(\bar{4}, 2) + 4\zeta^*(\bar{5}, 1), \\
\sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^4} (-1)^{n-1} &= -\frac{3}{2} \zeta(2) \zeta(3) \ln 2 + \frac{155}{32} \zeta(5) \ln 2 + \frac{7}{64} \zeta(6) + \frac{5}{4} \zeta^2(3) \\
&\quad + \zeta^*(\bar{4}, 2) + \frac{3}{2} \zeta^*(\bar{5}, 1).
\end{aligned}$$

On the other hand, by Cauchy product formula, we have

$$\text{Li}_2^2(x) = 2 \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(2)}{n^2} + 2 \frac{H_n}{n^3} \right\} x^n - 6\text{Li}_4(x). \quad (3.16)$$

Multiplying (3.16) by  $\frac{\ln(1-x)}{x}$ , and integrating over  $(0, x)$ . The result is

$$\text{Li}_2^3(x) = \sum_{n=1}^{\infty} \left\{ \frac{18}{n^5} - 6 \left[ \frac{\zeta_n(2)}{n^3} + 2 \frac{H_n}{n^4} \right] \right\} \left\{ x^n \ln(1-x) - \sum_{k=1}^n \frac{x^k}{k} - \ln(1-x) \right\}. \quad (3.17)$$

Taking  $x = -1$  in (3.17), using Example 2, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{L_n(1) \zeta_n(2)}{n^3} &= \frac{29}{8} \zeta(2) \zeta(3) \ln 2 - \frac{93}{32} \zeta(5) \ln 2 - \frac{1855}{128} \zeta(6) + \frac{17}{16} \zeta^2(3) - \zeta^*(\bar{5}, \bar{1}) \\ &\quad - \zeta^*(4, \bar{2}) - 4\zeta^*(\bar{4}, 2) - 8\zeta^*(\bar{5}, 1). \end{aligned} \quad (3.18)$$

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